J. Fluid Mech. (1975), *vol.* 70, *part* 1, *pp.* 37–57
Printed in Great Britain

A turbulence velocity scale for curved shear flows

By RONALD M. C. SO

Geophysical Fluid Dynamics Program, Rutgers University,
New Brunswick, New Jersey 08903

(Received 13 June 1974)

Assuming the turbulence length scale to be unaffected by streamline curvature, a turbulence velocity scale for curved shear flows is derived from the Reynolds-stress equations. Closure of the equations is obtained by using the scheme of Mellor & Herring (1973), and the Reynolds-stress equations are simplified by invoking the two-dimensional boundary-layer approximations and assuming that production of turbulent energy balances viscous dissipation. The resulting formula for the velocity scale has one free parameter, but this can be determined from data for non-rotating unstratified plane flows. Consequently there is no free constant in the derived expression. A single value of the constant is found to give good agreement between calculated and measured values of the velocity scale for a wide variety of curved shear flows.

The result is also applied to test the validity and extent of the analogy between the effects of buoyancy and streamline curvature. This is done by comparing the present result with that obtained by Mellor (1973). Excellent agreement is obtained for the range $-0{\cdot}21 \leqslant Ri_f \leqslant 0{\cdot}21$. Therefore the present result provides direct evidence in support of the use of a Monin–Oboukhov (1954) formula for curved shear flows as proposed by Bradshaw (1969).

1. Introduction

The importance of streamline curvature in a turbulent flow has been recognized by many investigators and various experimental efforts have been made to study its effects. For example, the effects of surface curvature on two-dimensional turbulent flows were investigated by Wattendorf (1935), Eskinazi & Yeh (1956), Giles, Hays & Sawyer (1966), Patel (1969), Ellis & Joubert (1974) and So & Mellor (1973, 1975). These studies show that curvature of the mean flow gives rise to a substantial change in the turbulent flow structure and an appreciable change in the measured mean velocity and wall shear stress. The most striking effect of curvature observed by So & Mellor (1973) in a turbulent boundary layer on a convex surface was that the shear stress vanishes in a region where the velocity gradient is still substantial. This seems to support the idea of a critical Richardson number beyond which turbulence cannot exist in convex curved shear flows.

The Coriolis effects on two-dimensional turbulent flows are equally striking and have been studied by Moore (1967), Halleen & Johnston (1967), Lezius &

Johnston (1971) and Johnston (1971). The latter found that for fully developed turbulent flows in a rotating channel increased rotation delays the transition to turbulence on the stabilized side of the channel. This finding is significant for the study of centrifugal compressors because according to Dean (1968), Coriolis effects contribute to the stabilization of suction surface layers and destablization of pressure surface layers on the blades.

Various attempts have been made to predict the observed effects of streamline curvature on shear-flow development. The approaches taken are quite different; however, the most widely accepted approach is based on Prandtl's mixing-length theory. Prandtl (1929) was the first researcher to extend the analogy between buoyancy and streamline-curvature effects to turbulent flow and he proposed accounting for either effect by multiplying the mixing length by a factor F which is a function of a dimensionless curvature or buoyancy parameter. He then used mixing-length arguments to estimate the behaviour of F and obtained

$$F = 1 - \beta \frac{U/r}{\partial U/\partial r},$$

where $\beta = \frac{1}{4}$ for curved flows. With the availability of more reliable curved-flow data (Wattendorf 1935; Schmidbauer 1936; Eskinazi & Yeh 1956), Prandtl's estimate of the curvature effects was found to be one order of magnitude lower than the observed effects, and a much larger β has to be used in order to account for the observed curvature effects. This introduces a certain arbitrariness into predictions for curved boundary layers.

Following Prandtl's line of reasoning, Bradshaw (1969) also drew an analogy between the effects of buoyancy and streamline curvature and argued heuristically that the non-conservative centrifugal forces exert their influence on the mixing length through the gradient Richardson number of the flow in question. He then used the analogy to apply meteorological data to curved shear flows and found that the use of a Monin–Oboukhov formula with a free constant considerably improves the agreement between prediction and experiment for boundary layers on curved surfaces. Therefore the F factor becomes

$$F = 1 - \beta Ri_B,$$

where Ri_B is the gradient Richardson number as defined by Bradshaw [given in equation (19*a*)]. However, Bradshaw found that he had to use a smaller β to account for flows on concave surfaces and for the effects of spanwise rotation. This approach was also adopted by such investigators as Rastogi & Whitelaw (1971) and Lezius & Johnston (1971), who found it necessary to change the value of β in order to obtain good agreement with data for curved jets and rotating channels. Therefore the arbitrariness inherent in Prandtl's original approach has not been eliminated by Bradshaw.

In a recent review of streamline-curvature effects, Bradshaw (1973) again argued in favour of this simple approach and obtained values of β that range from 2 to 7 depending on the type of shear flow considered. Bradshaw's approach is quite adequate for curved shear flows with very small values of δ/R; however, it fails badly to predict the data of So & Mellor (1973), where $\delta/R \sim 0{\cdot}1$. This indicates that a linear correction to the mixing length or eddy viscosity may not be adequate for curved shear flows with moderate to high values of δ/R. Besides, the F-factor correction was obtained from heuristic arguments and cannot be justified analytically.

The above approach is not very satisfactory because it involves more empiricism than is necessary and is also limited to flows with very small streamline curvature. To remedy the situation, So & Mellor (1972) proposed an eddy-viscosity function to account for effects of surface curvature. The eddy-viscosity function was obtained by closing the turbulent energy equations by suitably modelling the pressure–velocity correlation terms, the triple velocity correlation term and the viscous dissipation term and by invoking the boundary-layer approximations for a two-dimensional mean flow. There is one free parameter in the final expression; however, it can be determined from data for unstratified plane flows. As a result, there is no free constant in the eddy-viscosity function. So & Mellor (1972) then applied the eddy-viscosity function to calculate the boundary-layer development on curved surfaces and obtained good agreement with their measured results. In addition, the point beyond which turbulent shear stress could not be maintained was accurately predicted.

Later, Mellor (1973) used the same approach to simplify the one-point turbulent moment equations for velocity, temperature and pressure for a stratified planetary boundary layer and obtained good agreement with the Kansas data reported by Businger *et al.* (1971).

Here, the approach of So & Mellor (1972) will be extended to curved shear flows. The result will be used to test the validity and extent of the analogy between buoyancy and streamline-curvature effects by comparing it with the expression obtained by Mellor (1973). It will also be shown that the eddy-viscosity function can be recast in the form of an F-factor correction for the mixing length and that for small Richardson number the Monin–Oboukhov formula is recovered. Therefore, for the first time, evidence is obtained to support the use of the Monin–Oboukhov formula to account for the effects of curvature and rotation in a turbulent flow as suggested earlier by Bradshaw (1969).

2. The Reynolds-stress equations

In the analysis of So & Mellor (1972) it is assumed that $k\delta \ll 1$ (where $k(x)$ is the surface curvature, x is measured along the surface and δ is the boundary-layer thickness). This leads to a set of equations in which the exact metric influence of curvature is neglected. So (1975) showed that the neglect of the metric coefficient in the mean flow equations leads to extra terms in the von Kármán momentum integral and therefore introduces unnecessary errors into the solution of these equations. Also, the results using $k\delta \ll 1$ are not valid in regions of small curvature, which are commonly found near the leading edge of turbo-machinery blades. In the following analysis, $k\delta = O(1)$ will be considered so that the results may also be applied to the leading edge except in regions around the stagnation point, where the boundary-layer approximations fail.

Consider a flow rotating about a spanwise axis (i.e. the axis of rotation is normal to the plane of the mean rate of strain). In this case the 'centrifugal force' due to rotation can be absorbed into the mean pressure-gradient term if the flow is analysed with respect to a co-ordinate system fixed to the surface. The Reynolds-stress equations in tensor form are

$$\frac{\partial \overline{u_i u_j}}{\partial t}+\frac{\partial}{\partial x_k}\{U_k \overline{u_i u_j}+\overline{u_i u_j u_k}\} = -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k}-\overline{u_j u_k}\frac{\partial U_i}{\partial x_k}+2\{\epsilon_{ilm}\Omega_l \overline{u_m u_j}+\epsilon_{jlm}\Omega_l \overline{u_m u_i}\}$$
$$+\overline{p\frac{\partial u_i}{\partial x_j}}+\overline{p\frac{\partial u_j}{\partial x_i}}-\frac{\partial}{\partial x_k}\{\overline{pu_i}\,\delta_{jk}+\overline{pu_j}\,\delta_{ik}\}+\overline{u_i\frac{\partial \tau_{kj}}{\partial x_k}}+\overline{u_j\frac{\partial \tau_{ki}}{\partial x_k}}, \quad (1)$$

where x_i is the ith component of the Eulerian Cartesian co-ordinate, Ω_i, u_i and U_i are the ith components of the rotation vector and the fluctuating and mean velocities respectively, $p = p'/\rho$ is the fluctuating pressure, ρ is the fluid density, $\{\tau_{ij}\}$ is the viscous stress tensor and $\{\epsilon_{ijk}\}$ is the alternating tensor.

Equations (1) can be written in terms of the single-point double velocity correlations $\overline{u_i u_j}$ by appropriately modelling the terms $\overline{p\,\partial u_i/\partial x_j}$, $\overline{pu_i}$ and $\overline{u_i u_j u_k}$. The model terms proposed by Rotta (1951) and adopted by Mellor & Herring (1973) are

$$-\overline{p\frac{\partial u_i}{\partial x_j}} = \frac{1}{6}\frac{q}{l_1}\left(\overline{u_i u_j}-\delta_{ij}\frac{q^2}{3}\right), \quad (2a)$$

$$-\overline{pu_i} = \tfrac{1}{2}ql_2\,\partial q^2/\partial x_i, \quad (2b)$$

$$-\overline{u_i u_j u_k} = ql_3\left(\frac{\partial\overline{u_j u_k}}{\partial x_i}+\frac{\partial\overline{u_i u_k}}{\partial x_j}+\frac{\partial\overline{u_i u_j}}{\partial x_k}\right), \quad (2c)$$

where l_1, l_2 and l_3 are empirical length scales and are as yet undefined. The viscous terms in (1) can be written as

$$\overline{u_i\frac{\partial \tau_{kj}}{\partial x_k}}+\overline{u_j\frac{\partial \tau_{ki}}{\partial x_k}} = \nu\frac{\partial^2\overline{u_i u_j}}{\partial x_k^2}-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}, \quad (3)$$

and following Kolmogorov's (1941) hypothesis of local small-scale isotropy, the last term in (3) can be written as

$$2\nu\,\overline{\partial u_i\,\partial u_j/\partial x_k\,\partial x_k} = \tfrac{2}{3}\delta_{ij}q^3/\Lambda, \quad (4)$$

where $q = (\overline{u_i u_i})^{\frac{1}{2}}$ and Λ is the dissipation length scale.

Equations (2)–(4) are substituted into (1). The resultant equations in generalized tensor form are

$$\frac{\partial\overline{u_i u_j}}{\partial t}+U^k(\overline{u_i u_j})_{,k} = -\overline{u^k u_j}\,U_{i,k}-\overline{u^k u_i}\,U_{j,k}$$
$$+2\{\epsilon_{ilm}\,\Omega^l\overline{u^m u_j}+\epsilon_{jlm}\,\Omega^l\overline{u^m u_i}\}+[\tfrac{1}{2}ql_2(q^2{}_{,i})]_{,j}$$
$$+[\tfrac{1}{2}ql_2(q^2{}_{,j})]_{,i}+g^{kl}[ql_3(\overline{u_i u_{j,l}}+\overline{u_j u_{l,i}}+\overline{u_l u_{i,j}})]_{,k}$$
$$-\tfrac{1}{3}(q/l_1)\,[\overline{u_i u_j}-\tfrac{1}{3}g_{ij}q^2]+\nu g^{kl}(\overline{u_i u_j})_{,kl}-\tfrac{2}{3}g_{ij}q^3/\Lambda, \quad (5)$$

where $$q = (\overline{u^i u_i})^{\frac{1}{2}} \quad \text{and} \quad g_{ij} = \partial y^m\,\partial y^m/\partial x^i\,\partial x^j.$$

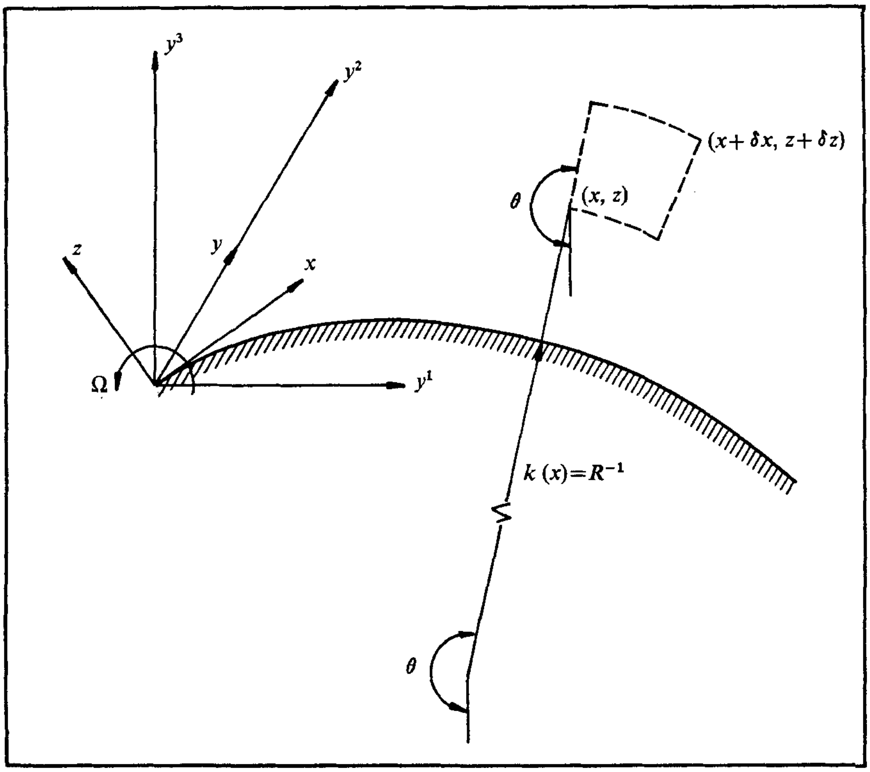

FIGURE 1. Curvilinear co-ordinate system.

A curvilinear orthogonal co-ordinate system fixed to the surface is considered and the rotation about the x^2 axis is assumed to have components $(0, \Omega, 0)$. If $\{x^i\} = (x, y, z)$ and $\{u_i\} = (u, v, w)$, then it can be shown that (figure 1)

$$\left.\begin{aligned} g_{11} = (1+kz)^2, \quad g_{22} = g_{33} = 1, \\ g_{ij} = 0, \quad i \neq j, \\ g \equiv \det(g_{ij}) = (1+kz)^2, \end{aligned}\right\} \tag{6}$$

$$\{g_{ij}\} = \begin{bmatrix} (1+kz)^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Since the instantaneous Coriolis force is always normal to the instantaneous velocity vector, Coriolis forces are not associated with direct energy production. However, the effects of rotation on production will be felt implicitly through the effects of Ω on the Reynolds stresses and mean shear. This can be seen by writing (5) in component form.

The component equations can be simplified by assuming that the flow is statistically steady and that production of turbulent energy balances viscous dissipation except in a very thin layer next to the surface and invoking the two-dimensional boundary-layer approximations. A model in which production balances dissipation was used by Bradshaw (1969) and is also a good approximation in the inner layer of the atmospheric boundary layer (Wyngaard & Coté 1971). Without loss of generality, the co-ordinate system can be oriented such that $\overline{vw} = 0$. If $2\Omega U$ is taken to be of the same order as $kU^2/(1+kz)$, then it follows from the boundary-layer approximations and the component equations for $\overline{uv}$ and $\overline{vw}$ that $\overline{uv} = 0$.† The resulting equations for $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ and $\overline{uw}$ are

$$\frac{1}{3}\frac{q}{l_1}\left(\overline{u^2}-\frac{q^2}{3}\right)+\frac{2}{3}\frac{q^3}{\Lambda}-2\frac{\tau}{\rho}\frac{\partial U}{\partial z}, \\ -\frac{2kU}{1+kz}\frac{\tau}{\rho}+4\Omega\frac{\tau}{\rho} = 0, \tag{8}$$

$$\frac{1}{3}\frac{q}{l_1}\left(\overline{v^2}-\frac{q^2}{3}\right)+\frac{2}{3}\frac{q^3}{\Lambda} = 0, \tag{9}$$

$$\frac{1}{3}\frac{q}{l_1}\left(\overline{w^2}-\frac{q^2}{3}\right)+\frac{2}{3}\frac{q^3}{\Lambda}+\frac{4kU}{1+kz}\frac{\tau}{\rho}-4\Omega\frac{\tau}{\rho} = 0, \tag{10}$$

$$\frac{1}{3}\frac{q}{l_1}\frac{\tau}{\rho}-\overline{w^2}\frac{\partial U}{\partial z}+\frac{kU}{1+kz}(2\overline{u^2}-\overline{w^2})-2\Omega(\overline{u^2}-\overline{w^2}) = 0, \tag{11}$$

where τ/ρ has been substituted for $-\overline{uw}$.

3. The turbulence velocity scale

The simplified Reynolds-stress equations (8)–(11) are algebraic in τ/ρ and can easily be solved in terms of k, Ω and the mean flow quantities. The solution is

$$\frac{\tau}{\rho} = l_0^2\left\{1-\beta\frac{\dfrac{kU}{1+kz}-\Omega}{\dfrac{\partial U}{\partial z}-\dfrac{kU}{1+kz}}\left(\frac{\dfrac{\partial U}{\partial z}+\dfrac{kU}{1+kz}-2\Omega}{\dfrac{\partial U}{\partial z}-\dfrac{kU}{1+kz}}\right)\right\}^{\frac{3}{2}}\left(\frac{\partial U}{\partial z}-\frac{kU}{1+kz}\right)^2, \tag{12}$$

where

$$l_0^2 = \{l_1\Lambda^{\frac{1}{3}}(1-6l_1/\Lambda)\}^{\frac{3}{2}}, \tag{12a}$$

$$\beta = \frac{72(l_1/\Lambda)}{1-6l_1/\Lambda}. \tag{12b}$$

A different form of (12) with $\Omega = 0$ and $1+kz \simeq 1$ was obtained by So & Mellor (1972).

In the absence of curvature and rotation, (12) reduces to

$$(\tau/\rho)_0 = l_0^2(\partial U/\partial z)_0^2, \tag{13}$$

where the subscripts zero denote properties of a corresponding non-rotating plane flow. It can be seen that (13) is the familiar Prandtl expression for the turbulent shear stress and l_0 has the meaning of a mixing length. On the other hand, if an

† For three-dimensional flows or for flows in which the plane of surface curvature, the plane of rotation and the plane of the mean shear do not coincide, $\overline{uv}$ and $\overline{vw}$ are not zero. A discussion of three-dimensional effects is given by Bradshaw (1973). Here, only two-dimensional flows or flows in which all three planes coincide are considered.

effective viscosity ν_{e0} is defined as $\nu_{e0} \equiv q_0 l_0$, then (13) gives the turbulence velocity scale as

$$q_0 = l_0(\partial U/\partial z)_0 \tag{14}$$

and l_0 can be interpreted as the turbulence length scale.

Since (12) is derived from the Reynolds-stress transport equations, it is reasonable to assume that the streamline-curvature effects thus deduced affect only the turbulence velocity scale and not the length scale. Therefore an effective viscosity $\nu_e = ql_0$ can be defined for curved shear flows and the shear stress can then be written as

$$\frac{\tau}{\rho} = \nu_e\left(\frac{\partial U}{\partial z} - \frac{kU}{1+kz}\right). \tag{15}$$

Note that (15) is chosen such that it has the same form as the corresponding equation for laminar shear flows.

If the gradient Richardson number Ri is defined as

$$Ri = \frac{\text{typical body force}}{\text{typical inertia force}},$$

then it can easily be shown that for the curved shear flow in question

$$Ri = S(1+S), \tag{16a}$$

where

$$S = \left(\frac{2kU}{1+kz} - 2\Omega\right)\Big/\left(\frac{\partial U}{\partial z} - \frac{kU}{1+kz}\right). \tag{16b}$$

With the help of (14)–(16) and $\nu_e = ql_0$, the velocity-scale formula for curved shear flow can be deduced as

$$\frac{q}{q_0} = \frac{\nu_e}{\nu_{e0}} = \frac{(1-\frac{1}{2}\beta Ri)^{\frac{1}{2}}}{1+\frac{1}{2}S_c}\frac{\partial U/\partial z}{(\partial U/\partial z)_0}, \tag{17}$$

where

$$S_c = \left(\frac{2kU}{1+kz}\right)\Big/\left(\frac{\partial U}{\partial z} - \frac{kU}{1+kz}\right). \tag{18}$$

For non-rotating curved flows, $S = S_c$ and $Ri = Ri_c = S_c(1+S_c)$. However, from heuristic arguments, Bradshaw (1969) defined the gradient Richardson number as

$$Ri_B = 2S_B(1+S_B), \tag{19a}$$

where

$$S_B = \left(\frac{kU}{1+kz}\right)\Big/\frac{\partial U}{\partial z}. \tag{19b}$$

The difference between Ri_B and Ri_c is negligible when $S_B \ll 1$; however, it becomes significant when $S_B \sim 1$. Therefore it is important to retain the curvature term as well as the gradient term in the definition of Ri, especially in cases where $k\delta = O(1)$.

For rotating plane flows, (16*a*) gives $Ri = Ri_R = S_R(1+S_R)$, where S_R is given by

$$S_R = -\frac{2\Omega}{\partial U/\partial z}, \tag{20}$$

and the velocity-scale formula is again given by (17) but with $S_c \equiv 0$.

Alternatively, (17) can be written in terms of the flux Richardson number Ri_f, which can be defined as

$$Ri_f = -\frac{w\text{-component energy production due to body force}}{u\text{-component energy production}}.$$

From (8), (10) and (16*b*), Ri_f can be written as

$$Ri_f = S/(1+S) \tag{21}$$

and with the help of (16*a*) and (21), (17) becomes

$$\frac{q}{q_0} = \frac{[1-\frac{1}{2}\beta Ri_f(1-Ri_f)^{-2}]^{\frac{1}{4}}}{1+\frac{1}{2}S_c}\frac{\partial U/\partial z}{(\partial U/\partial z)_0}. \tag{22}$$

Therefore, either (17) or (22) can be used to estimate the velocity scale for curved shear flows depending on which is more convenient.

4. Determination of β

Equations (17) and (22) are completely defined except for the parameter β, which is defined in (12*b*). So & Mellor (1972) have shown that l_1/Λ can be easily determined from (8)–(11) by considering the wall-law region of non-rotating flows on plane surfaces. From the data of Laufer (1954) and Klebanoff (1955), So & Mellor (1972) determined $q/u_\tau \simeq 2{\cdot}4$ and therefore obtained a value of 0·04 for l_1/Λ and ~ 4 for β. On the other hand, if q/u_τ is taken to be 2·3 [which amounts to an error of $\sim 4\,\%$ in the interpolation of the data of Laufer (1954) and Klebanoff (1955)], the corresponding values for l_1/Λ and β are 0·053† and ~ 6 respectively. Previously, So & Mellor (1972) had found that $\beta = 4$ gives a better prediction of their curved boundary-layer flows, but as will be seen later, $\beta = 6$ gives a better overall prediction of all kinds of curved-flow data, rotating-channel data and meteorological data.

5. Comparison with measurements

Experimental data on two-dimensional rotating curved flows are practically non-existent. The known exceptions are the rotating-cylinder results of Parr (1963); however, he reported no measured shear-stress profiles. In view of this, (17) will be compared with measurements obtained from non-rotating curved flows and rotating plane flows. The data chosen are those obtained by Wattendorf (1935), Schubauer & Klebanoff (1951), So & Mellor (1973) and Halleen & Johnston (1967).

The result given in (12) can be written alternatively in terms of a mixing length l such that

$$\frac{\tau}{\rho} = l^2\left(\frac{\partial U}{\partial z} - \frac{kU}{1+kz}\right)^2, \tag{23a}$$

and

$$l^2/l_0^2 = (1-\tfrac{1}{2}\beta Ri)^{\frac{1}{2}}. \tag{23b}$$

† A value of 0·052 was used by Mellor (1973), who found good agreement with the atmospheric data of Businger *et al.* (1971).

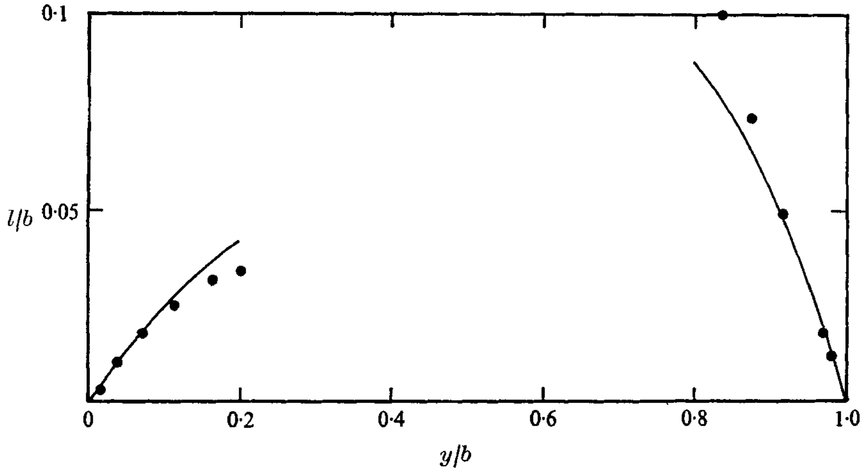

FIGURE 2. Comparison of calculated and measured mixing length in a fully developed curved channel flow. ●, Wattendorf's (1935) data; ——, present calculation.

Then (17) and (23*b*) together give

$$\frac{l}{l_0} = \left[\frac{q(\partial U/\partial z)_0}{q_0 \partial U/\partial z}(1+\tfrac{1}{2}S_c)\right]^{\frac{1}{2}} = (1-\tfrac{1}{2}\beta Ri)^{\frac{1}{4}}. \tag{24}$$

In view of the fact that most data are reported in the form l/l_0, comparison of measured data will be made with (24) instead of (17). Also, it should be noted that Ri_c and Ri_R are calculated from graphical differentiation of measured mean velocity profiles; therefore, estimates of uncertainty of $\pm 15\,\%$ are representative of the overall accuracy of the results shown.

In evaluating S_c from Wattendorf's (1935) data, the value of $\partial U/\partial z$ near the wall is calculated using the method suggested by Wattendorf. In the central core of the channel, $\partial U/\partial z$ is calculated using graphical differentiation. The l_0 used are those calculated by Wattendorf in the straight section of his channel. l/b is plotted *vs.* z/b in figure 2, where b is the channel width. It can be seen that (24) with $\beta = 6$ and $Ri = Ri_c$ correlates well with the data near the wall. Beyond $z/b = 0{\cdot}1$, $\frac{1}{2}\beta Ri_c > 1$, which is a consequence of the large Ri_c in this region (figure 3). The large magnitude of Ri_c arises entirely from its normalization by $\partial U/\partial z - kU/(1+kz)$ and does not reflect the magnitude of the curvature effects. A similar problem exists for rotating flow; however, Lezius & Johnston (1971) suggested a redefinition of Ri_R so that a smooth distribution of $\partial U/\partial z$ (with $\partial U/\partial z = 0$ at $\tau/\rho = 0$) is obtained. Adopting this approach for curved flows, Ri_c is redefined as shown by the broken line in figure 3 and also the results obtained for l/b for $0{\cdot}1 < z/b < 0{\cdot}2$ and $0{\cdot}8 < z/b < 0{\cdot}9$ are plotted in figure 2. Again good agreement with Wattendorf's (1934) data is obtained.

The measured data of So & Mellor (1973) at $x = 71$ in. are chosen for comparison. The mixing length l is calculated from the measured $-\overline{uw}$ and mean velocity profiles, while l_0 is calculated from the corresponding measurements obtained at $x = 24$ in., at which there was a flat-plate equilibrium boundary layer. The results are shown in figure 4 together with two curves ($\beta = 4$ and 6)

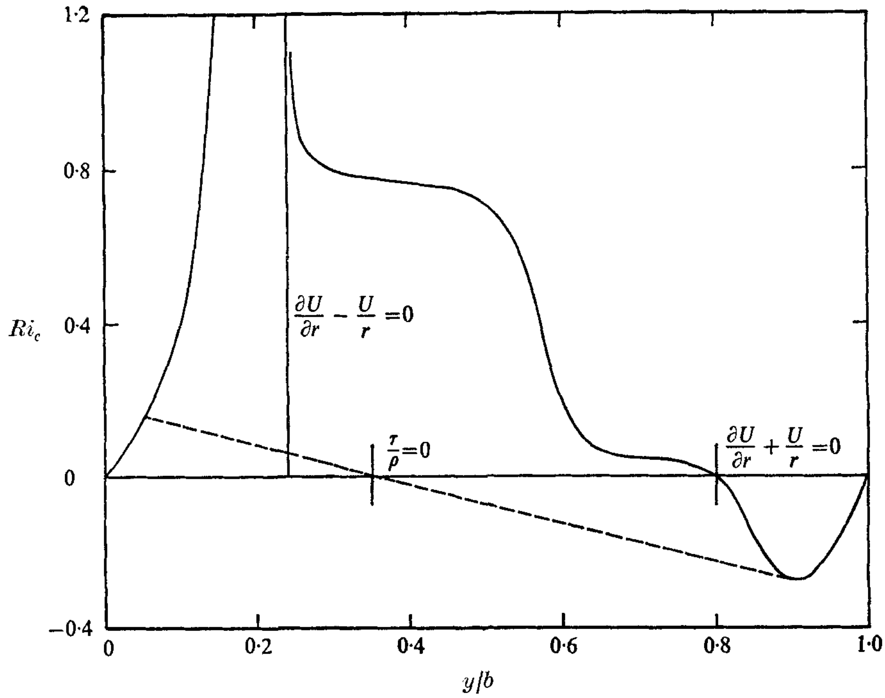

FIGURE 3. Distribution of gradient Richardson number Ri_c across the channel in a fully developed curved-channel flow. ----, redefinition of Ri_c for purpose of calculation.

calculated from (24). Previously, So & Mellor (1972) had shown that $\beta = 4$ gives a better prediction of the boundary-layer development on convex surfaces; however the present results show that the measured data actually correlate better with the $\beta = 6$ curve. The critical Richardson number beyond which turbulence cannot exist determined from the data at $x = 71$ in. and $y/\delta \simeq 0{\cdot}4$ is 0·35.† This compares well with the value of 0·33 calculated from (24) with $\beta = 6$.

The other well-authenticated test case with a prolonged region of surface curvature is the separated-flow experiment of Schubauer & Klebanoff (1951). The aerofoil used has a radius of curvature of 31 ft downstream of the pressure minimum ($x \simeq 18$ ft), where $\delta \simeq 3$ in. The measurements at $x = 23{\cdot}5$ and 25 ft are chosen for comparison while l_0 is calculated from the measurements at $x = 17{\cdot}5$ ft, at which there was a flat-plate boundary layer. The results are shown in figure 5. It can be seen that the effect of surface curvature on the velocity scale is small; however, it is significant enough to have an appreciable effect on the boundary-layer development (Bradshaw 1969). In spite of the large scatter ($\pm 20\,\%$) seen for $Ri_c < 0{\cdot}02$, it can be said that the value $\beta = 6$ determined from data for unstratified plane flows is independent of the effects of pressure gradient.

The data of Halleen & Johnston (1967) have previously been analysed by

† So & Mellor (1973) calculated Ri_B and obtained a value of 0·30 for the critical Richardson number. The difference is $\sim 15\,\%$ and tends to illustrate the importance of defining the Richardson number as Ri_c rather than as Ri_B.

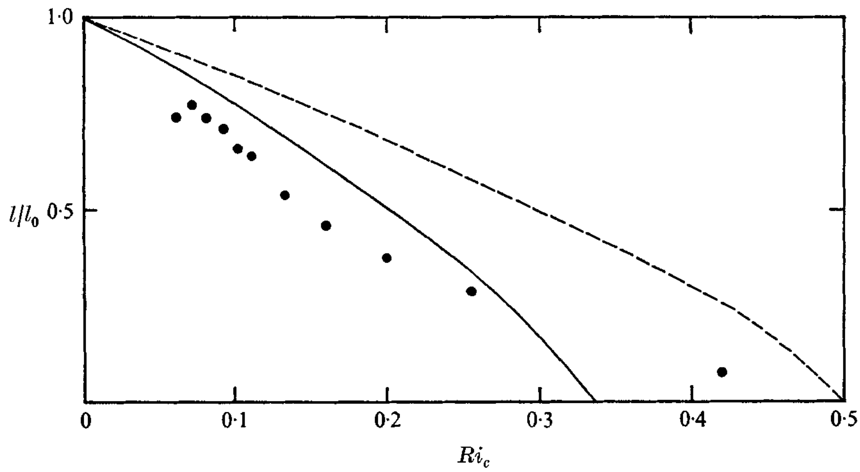

FIGURE 4. Comparison of calculated and measured mixing-length ratio in a boundary layer on a convex surface. ●, So & Mellor's (1973) data. Present calculations: ——, $\beta = 6$; ----, $\beta = 4$.

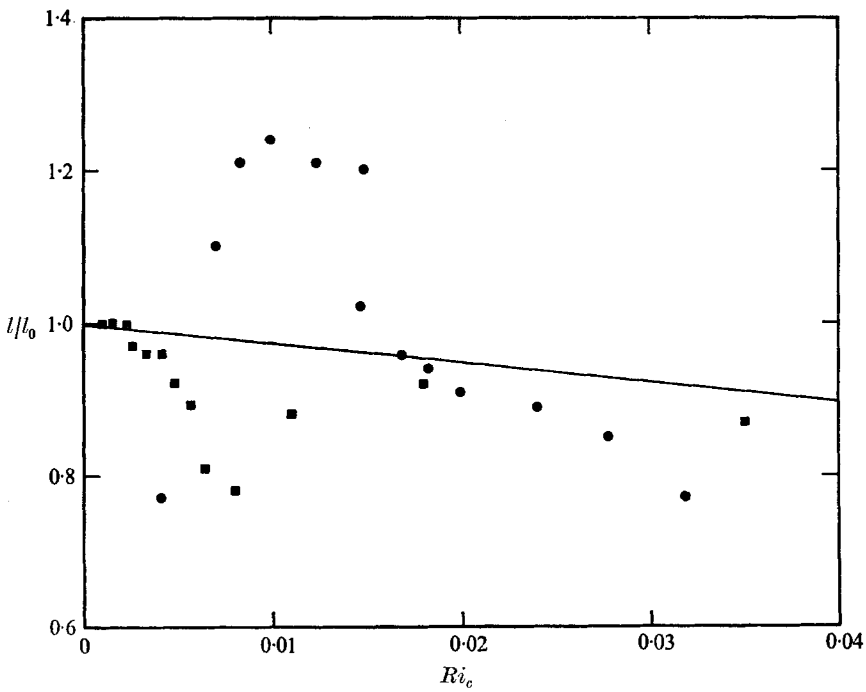

FIGURE 5. Comparison of calculated and measured mixing-length ratio in a curved boundary layer near separation. Schubauer & Klebanoff's (1951) data: ■, $x = 23{\cdot}5$ ft; ●, $x = 25$ ft. ——, present calculation.

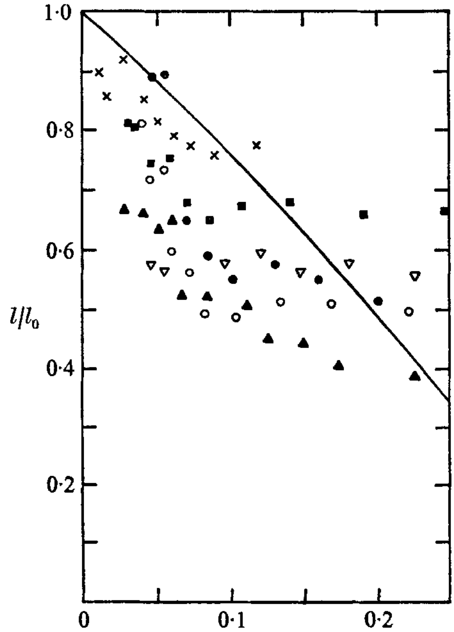

FIGURE 6

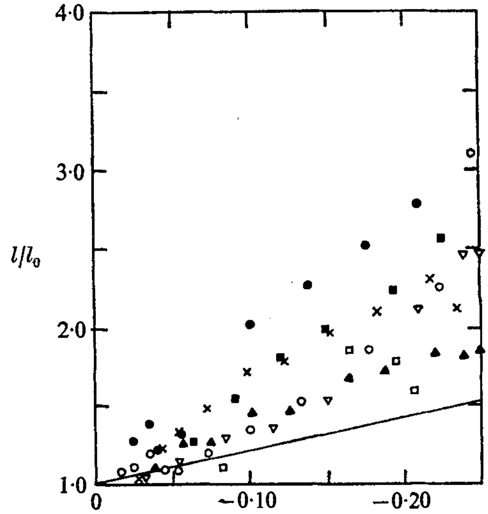

FIGURE 7

FIGURE 6. Comparison of calculated and measured mixing-length ratio on the trailing side of a fully developed rotating channel flow. Lezius & Johnston's (1971) data: ▽, $Ro = 0{\cdot}069$, $Re = 1{\cdot}14 \times 10^4$; ×, $Ro = 0{\cdot}027$; ■, $Ro = 0{\cdot}042$; ●, $Ro = 0{\cdot}056$; ○, $Ro = 0{\cdot}068$; ▲, $Ro = 0{\cdot}081$, $Re = 3{\cdot}54 \times 10^4$. ——, present calculation. $Ro = 2\Omega D/U_m$, $Re = 2U_m D/\nu$, where U_m is the average mean velocity and D is the channel half-width.

FIGURE 7. Comparison of calculated and measured mixing-length ratio on the leading side of a fully developed rotating channel flow. Lezius & Johnston's (1971) data: ▽, $Ro = 0{\cdot}069$; □, $Ro = 0{\cdot}117$, $Re = 1{\cdot}14 \times 10^4$; ×, $Ro = 0{\cdot}027$; ■, $Ro = 0{\cdot}042$; ●, $Ro = 0{\cdot}056$; ○, $Ro = 0{\cdot}068$; ▲, $Ro = 0{\cdot}081$, $Re = 3{\cdot}54 \times 10^4$. ——, present calculation. See figure 6 for definition of Ro and Re.

Lezius & Johnston (1971), therefore the present calculations using (24) with $\beta = 6$ and $Ri = Ri_R$ are compared with the data reported in Lezius & Johnston (1971). The results are given in figures 6 and 7 and show that (24) correlates well with the measured data on the stabilized side of the channel but gives a value that is consistently too low on the unstable side. As shown by Lezius & Johnston (1971), this would lead to an underprediction of the mean velocity, but would not have an appreciable effect on the wall shear stress.

6. Examination of the curvature/buoyancy analogy

There is a close analogy between streamline curvature and buoyancy in laminar flows (Görtler 1959); however, the analogy is not quite so close in turbulent flows. This is evident from the early work of Prandtl (1929) and the experimental measurements of Wattendorf (1935), Schmidbauer (1936) and Eskinazi & Yeh (1956). Further evidence is also provided by the experiment of Johnson (1959), who found the correlation coefficient of temperature and velocity fluctuations to be approximately 0·7 in a shear flow over a heated flat plate. Nevertheless, Bradshaw (1969, 1973) made use of the idea to apply meteorological data to curved turbulent shear flows and found good agreement between calculated and measured boundary-layer development on curved surfaces with a radius of curvature large compared with the boundary-layer thickness.

The assumption that the curvature/buoyancy analogy is valid is implicit in the work of Prandtl and Bradshaw; however, the validity and extent of this approximation have not been carefully examined. Recently, Mellor (1973) used a model in which production balances dissipation to analyse a stratified boundary-layer flow and obtained an analytic expression for the shear stress $-\overline{uw}$. In view of this, an attempt will be made to assess the validity and extent of the curvature/buoyancy analogy by comparing the present result for $-\overline{uw}$ with that obtained by Mellor (1973).

Following Monin & Oboukhov (1954), the non-dimensional quantity ϕ_M is defined by

$$\phi_M = \frac{\kappa z}{u_\tau}\frac{\partial U}{\partial z}, \tag{25a}$$

where κ is the von Kármán constant and $u_\tau = (\tau/\rho)^{\frac{1}{2}}$ is the friction velocity. Now, ϕ_M can be interpreted as the ratio of two mixing lengths, i.e.

$$\phi_M = l_0/l, \tag{25b}$$

where $l_0 = \kappa z$ is the mixing length of the atmospheric surface layer in unstratified air and l is the corresponding value in a stratified medium. With the help of (16a), (21) and (23), (12) can be written as

$$l/l_0 = [1-\tfrac{1}{2}\beta Ri_f/(1-Ri_f)^2]^{\frac{3}{4}}. \tag{26}$$

If a formal analogy is drawn between streamline curvature and buoyancy, (25b) and (26) together give

$$\phi_M = l_0/l = S_M^{-\frac{1}{2}}, \tag{27a}$$

where

$$S_M = [1-\tfrac{1}{2}\beta Ri_f/(1-Ri_f)^2]^{\frac{3}{2}} \tag{27b}$$

and Ri_f is now the flux Richardson number for stratified flows, which is defined as the ratio of negative buoyant production to shear production, i.e.

$$Ri_f = \frac{\alpha g\overline{w\theta}}{\overline{uw}\,\partial U/\partial z}. \tag{28}$$

In (28), θ is the potential temperature, $-\overline{w\theta}$ is the thermometric heat flux, $\alpha = -(\partial\rho/\partial T)_p/\rho$ is the coefficient of thermal expansion, T is the mean temperature and g is the acceleration due to gravity.

The result obtained by Mellor (1973) for $-\overline{uw}$ in a stratified planetary boundary layer can be conveniently written as

$$\tilde{\phi}_M = \tilde{S}_M^{-\frac{1}{2}}, \tag{29a}$$

where

$$\tilde{S}_M = (1-Ri_f)^{\frac{1}{2}}\left[\frac{(1-3{\cdot}686\Gamma)(1-2{\cdot}700\Gamma)}{1-3{\cdot}006\Gamma}\right]^{\frac{3}{2}}, \quad \Gamma = \frac{Ri_f}{1-Ri_f}. \tag{29b,c}$$

The flux Richardson number Ri_f is again defined by (28). In deducing (29) from Mellor's results, use is made of (25 *a*), it is assumed that $l_0 = \kappa z$ and $-\overline{uw} = u_\tau^2$ in the constant-flux layer and the constants A_1, A_2, B_1, B_2 and C are taken to be 0·78, 0·79, 15·0, 8·0 and 0·056 respectively as given by Mellor (1973). Equation (29 *b*) with the constants restored is also given by Mellor & Yamada (1974). If $\zeta \equiv z/L$, where $L = u_\tau^3/\kappa\alpha g(-\overline{w\theta})$ is the Monin–Oboukhov length scale, then it can easily be shown that $\zeta = \tilde{\phi}_M Ri_f$. This result presented in the form $\tilde{\phi}_M(\zeta)$ has been compared with the constant-flux data of Businger *et al.* (1971) by Mellor (1973) and good agreement was obtained.

The validity and extent of the curvature/buoyancy analogy can now be assessed by comparing (27) with (29). Taking $\beta = 6$, (27) gives a critical flux Richardson number $Ri_{f\mathrm{cr}} = 0{\cdot}209$ beyond which turbulence cannot exist. Similarly, (29) gives $Ri_{f\mathrm{cr}} = 0{\cdot}213$. Therefore the analogy can be considered valid as far as estimating the limit of turbulent energy production in a stably stratified boundary layer is concerned. The accuracy of this estimate compared with measured values will be discussed later.

The behaviour of ϕ_M and $\tilde{\phi}_M$ in the limits $Ri_f \to 0$ and $Ri_f \to -\infty$ is given by

$$\phi_M \sim 1 + 2{\cdot}25\,Ri_f, \quad \tilde{\phi}_M \sim 1 + 2{\cdot}785\,Ri_f \quad \text{as} \quad Ri_f \to 0, \tag{30 a, b}$$

$$\phi_M \sim 1, \quad \tilde{\phi}_M \sim 0{\cdot}333(-Ri_f)^{-\frac{1}{4}} \quad \text{as} \quad Ri_f \to -\infty, \tag{31 a, b}$$

which clearly shows that the curvature/buoyancy analogy is not valid for large values of $-Ri_f$. A plot of ϕ_M^{-1} and $\tilde{\phi}_M^{-1}$ *vs.* Ri_f is given in figure 8. It can be seen that for small positive values of Ri_f the functions ϕ_M^{-1} and $\tilde{\phi}_M^{-1}$ are identical. Therefore the analogy is most valid for stable flows and the range of Ri_f in which it can be applied is $-0{\cdot}21 \leqslant Ri_f \leqslant 0{\cdot}21$.

If an effective viscosity K_M and an effective conductivity K_H are defined for a stratified shear flow such that

$$\tau/\rho = -\overline{uw} = K_M\,\partial U/\partial z, \tag{32 a}$$

$$H = -\overline{w\theta} = K_H\,\partial\Theta/\partial z, \tag{32 b}$$

where Θ is the mean potential temperature, and if a non-dimensional quantity

$$\phi_H = \frac{\kappa z u_\tau}{H}\frac{\partial\Theta}{\partial z} \tag{33}$$

is introduced, then it can be seen from (25 *a*), (32) and (33) that

$$\frac{\phi_H}{\phi_M} = \frac{K_M}{K_H} = Pr_t, \tag{34}$$

where Pr_t is the turbulent Prandtl number. Combining (27) and (34) the following result is obtained:

$$\phi_H = Pr_t S_M^{-\frac{1}{2}}. \tag{35}$$

Mellor's result for a stratified flow can be written as

$$\tilde{\phi}_H = \tilde{S}_M^{\frac{1}{2}} \tilde{S}_H^{-1}, \tag{36 a}$$

where
$$S_H = 1{\cdot}351(1 - Ri_f)^{\frac{1}{2}}(1 - 3{\cdot}686\Gamma)\left[\frac{(1-3{\cdot}686\Gamma)(1-2{\cdot}700\Gamma)}{1-3{\cdot}006\Gamma}\right]^{\frac{1}{2}}. \tag{36 b}$$

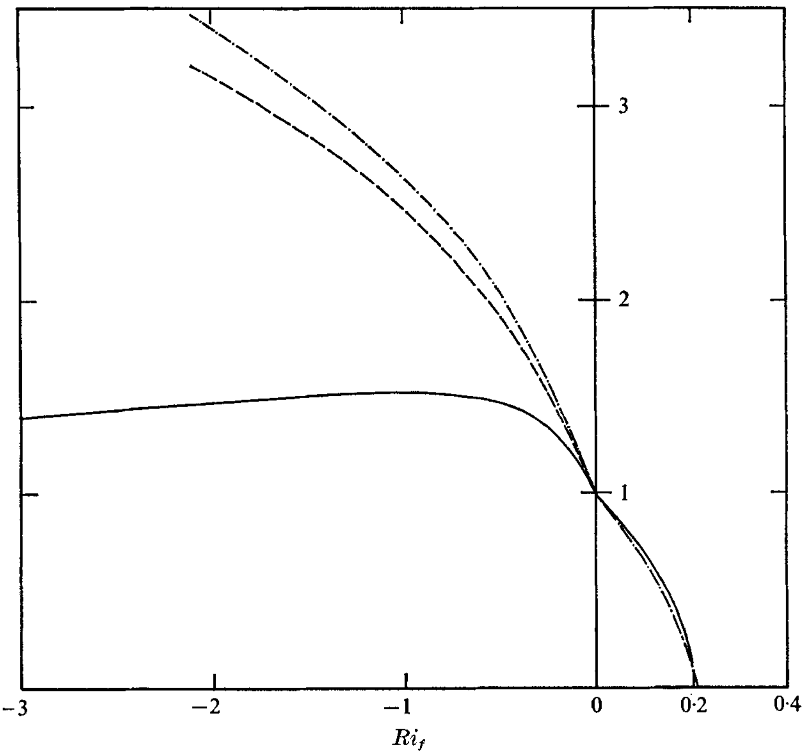

FIGURE 8. Comparison of the curvature/buoyancy analogy with the analytic results of Mellor (1973). ——, ϕ_M^{-1}; – –, $\tilde{\phi}_M^{-1}$; —·—, $\tilde{\phi}_H^{-1}/1{\cdot}351$. Note the change of scale in Ri_f.

Again, the behaviour of $\tilde{\phi}_H$ as $Ri_f \to 0$ and $Ri_f \to -\infty$ is given by

$$\tilde{\phi}_H \sim \begin{cases} 0{\cdot}740(1+3{\cdot}025\,Ri_f) & \text{as} \quad Ri_f \to 0, \qquad (37a) \\ 0{\cdot}230(-Ri_f)^{-\frac{1}{4}} & \text{as} \quad Ri_f \to -\infty. \qquad (37b) \end{cases}$$

If Pr_t is chosen to be 0·74, it can be seen that the curvature/buoyancy analogy can also be applied to estimate the heat flux in a stratified shear layer (figure 8) and the range of Ri_f in which the analogy is valid is again seen to be

$$-0{\cdot}21 \leqslant Ri_f \leqslant 0{\cdot}21.$$

For $Ri_f < 0$, the agreement between ϕ_M and $\tilde{\phi}_M$ and between ϕ_H and $\tilde{\phi}_H$ is rather poor except in the range $Ri_f > -0{\cdot}20$. On the other hand, the agreement between $\tilde{\phi}_M$ and $\tilde{\phi}_H$ and the measurements of Businger *et al.* (1971) is only marginal for $Ri_f < -0{\cdot}20$. In view of this, the part of the range $Ri_f < -0{\cdot}20$ in which the analogy is valid will be established by direct comparison of the present results with the measurements of Businger *et al.* (1971). If $Ri = Ri_f(1-Ri_f)^{-2}$ is substituted into (26) and an analogy is drawn between streamline curvature and buoyancy, then it can be shown that

$$S_M = [1-\tfrac{1}{2}\beta Ri]^{\frac{3}{2}}, \qquad (38)$$

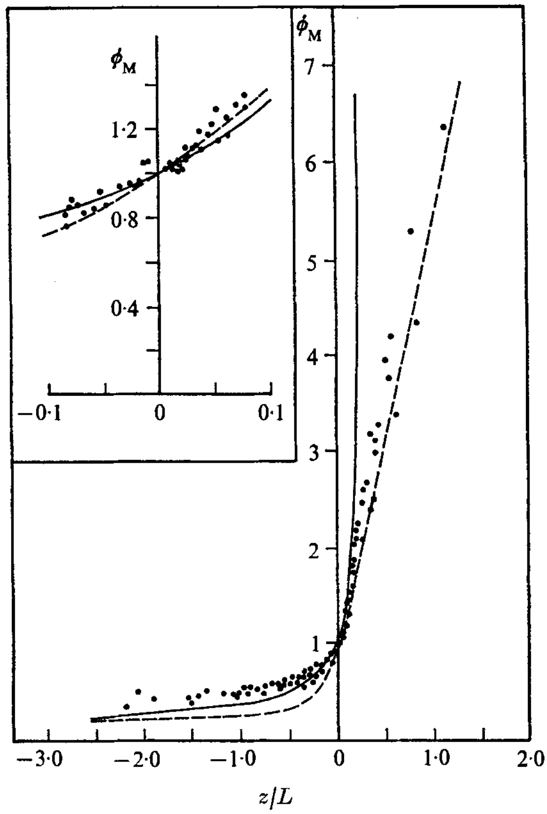

FIGURE 9. Comparison of calculated and measured dimensionless wind shear. ●, data of Businger *et al.* (1971); ——, present calculation; – – –, Mellor's (1973) calculation.

where *Ri* is now the gradient Richardson number for stratified flows and can be written as

$$Ri = \frac{g}{T}\frac{\partial\Theta/\partial z}{(\partial U/\partial z)^2}. \tag{39}$$

According to similarity theory, ϕ_M and ϕ_H are universal functions of z/L (Lumley & Panofsky 1964, p. 106), therefore Ri is also a function of z/L. For unstable stratification, it has been suggested by Pandolfo (1966) and Businger (1966) that $Ri = z/L$ is a good approximation for Ri, and the Kansas data analysed by Businger *et al.* (1971) essentially substantiate this conclusion. For stable stratification, however, no such simple relation exists between Ri and z/L. But for small z/L, the analysis of Businger *et al.* (1971) showed that $Ri \simeq 0{\cdot}74(z/L)$. The simple expression $Ri = z/L$ is therefore a good first approximation for the present purpose. With this substitution, (38) becomes

$$S_M = [1-\tfrac{1}{2}\beta(z/L)]^{\frac{1}{2}}. \tag{40}$$

This expression is used to calculate ϕ_M and ϕ_H from (27 *a*) and (35) and the results are shown in figures 9 and 10. Shown also in these figures are the measurements of Businger *et al.* (1971) and the calculations of Mellor (1973) given by (29 *a*) and

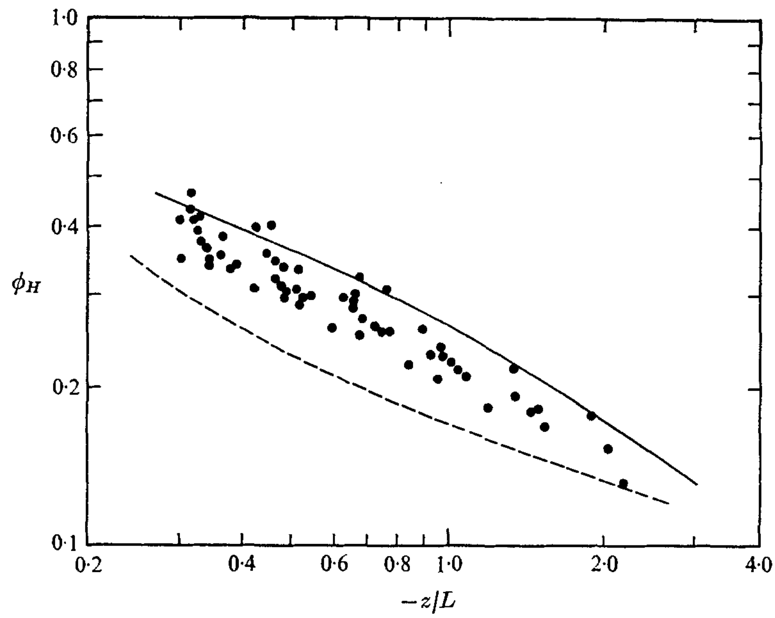

FIGURE 10. Comparison of temperature variance under unstable conditions. ●, data of Businger *et al.* (1971); ——, present calculation; – – –, Mellor's (1973) calculation.

(36 *a*), while the behaviour of ϕ_M and $\tilde{\phi}_M$ for small z/L is included as an inset in figure 9. The results show that, for $-2 < z/L < 0$, ϕ_M and ϕ_H correlate better with the measurements of Businger *et al.* (1971) than $\tilde{\phi}_M$ and $\tilde{\phi}_H$. Since, for small values of S, $Ri \simeq Ri_f$, the curvature/buoyancy analogy can therefore be considered to be valid in the range $-2 < Ri \leqslant 0{\cdot}21$. However, more data are needed to provide added evidence to support this claim.

7. Discussion

Most previous work on the effects of surface curvature or spanwise rotation is described using an F factor for the mixing length. In order to compare those results with the present one, the F factor is rewritten in the form of a correction factor for the turbulence velocity scale, which for small values of Ri is most conveniently given as

$$\frac{q(\partial U/\partial z)_0}{q_0(\partial U/\partial z)} = \frac{l^2}{l_0^2} = 1 - 2n\beta \begin{Bmatrix} Ri_B \\ Ri_R \end{Bmatrix}, \tag{41}$$

where n and β are free constants. A number of different values for n and β have been used by researchers in the past. Usually, they were selected to fit best the experimental flow under consideration. For example, Giles *et al.* (1966) chose $n = 1$ and $\beta = 3$ for curved wall jets while Rastogi & Whitelaw (1971) selected $n = \frac{1}{2}$ and $\beta = 4{\cdot}5$ for curved boundary-layer flows and wall jets and Bradshaw (1969) used $n = 1$ with $\beta = 4{\cdot}5$ and 7 in his investigation of boundary-layer flows on convex and concave surfaces respectively. However, for rotating-channel flow, Bradshaw (1969) found it necessary to select $n = 1$ with $\beta = 4$ and 2 for the flow on the stable and unstable sides of the channel respectively, but Lezius & Johnston (1971) found that $n = 1$ with $\beta = 6$ gave a best overall fit to the data of Halleen & Johnston (1967). In general, the values chosen for n and β give a coefficient that lies in the range $4 \leqslant 2n\beta \leqslant 14$.

In the case of curved or rotating flows, (17) can be simplified for small Ri_c or Ri_R, and the result is

$$\frac{q(\partial U/\partial z)_0}{q_0\,\partial U/\partial z} = 1 - \begin{Bmatrix} \frac{1}{4}(3\beta+2) \\ \frac{3}{4}\beta \end{Bmatrix} \begin{Bmatrix} Ri_c \\ Ri_R \end{Bmatrix}. \tag{42}$$

The coefficients obtained by taking $\beta = 4$ or 6 are 3·5 or 3 and 5 or 4·5 respectively. It can be seen that the latter values fall in the range $4 \leqslant 2n\beta \leqslant 14$ and compare most favourably with the value of 4 found by Bradshaw (1969) to give the best straight-line fit with available data on curved and rotating flows over the range $-0{\cdot}5 < Ri_B, Ri_c < 0$. Although a comparison of (17) with curved-jet data has not been made, the results of Giles *et al.* (1966) and Rastogi & Whitelaw (1971) provide the most favourable evidence supporting the extension of (17) to curved-jet flows. Since both Coriolis and 'centrifugal' forces depend on the u component of velocity, one would intuitively expect the same β to apply to both rotating and curved flows. The present results show that this is indeed true. Therefore, it can be seen that $\beta = 6$, determined from data for non-rotating unstratified plane flows, is the correct constant to use in (17) for a wide variety of curved and rotating flows and that β is independent of the effects of body forces and pressure gradient.

The results given in (42) provide, for the first time, direct evidence supporting the use of a Monin–Oboukhov formula to correct for the effects of surface curvature and spanwise rotation on the turbulence length or velocity scale. However, for rotating flow on curved surfaces, the Monin–Oboukhov formula has to be modified by the factor $(1+\frac{1}{2}S_c)^{-1}$ because a small Ri does not necessarily imply that Ri_c is also small. In order to verify this, experimental work on two-dimensional rotating flow on curved surfaces has been planned.

The collapse of turbulence is a common phenomenon in a stably stratified atmosphere and its occurrence is governed by a critical Richardson number Ri_{cr}, which was originally determined by Richardson (1920), who assumed that $K_M = K_H$ and found $Ri_{cr} = 1$. On the other hand, if $K_M = K_H$ is not assumed but the stratified flow is considered to be statistically steady and transport of turbulent energy by diffusion and through advection by the mean flow is considered negligible, then the turbulent energy equation can be written as

$$1 - Ri_f = \text{dissipation}.$$

Therefore, the critical flux Richardson number $Ri_{f\,cr}$ beyond which turbulence cannot exist also has an absolute upper limit of one (the turbulent dissipation of energy cannot change sign). Since viscous dissipation is always present in any real flow and serves as a turbulent energy sink, it is reasonable to expect that $Ri_{f\,cr}$ or Ri_{cr} will have a value less than one. Townsend (1958) found that, if the turbulent intensity is also required to satisfy the equation for the intensity of the temperature fluctuations, then $Ri_{f\,cr} < \frac{1}{2}$. This result is supported by the semi-empirical analysis of Arya (1972), who found that $Ri_{f\,cr}$ ranges between 0·15 and 0·25, and by the experimental evidence of Businger *et al.* (1971), who reported that $Ri_{f\,cr} = 0{\cdot}25$. The recent analysis of Mellor (1973) gives $Ri_{f\,cr} = 0{\cdot}213$ while the present analysis, together with the assumption of a curvature/buoyancy analogy, gives $Ri_{f\,cr} = 0{\cdot}209$. In view of this, further evidence in support of the validity of the curvature/buoyancy analogy in a stably stratified flow has been obtained.

The critical flux Richardson number can be interpreted as corresponding to the total production of turbulent energy in the boundary layer becoming very small while dissipation continues with the result that the whole level of turbulence decreases. For curved flows, the simplified turbulent energy equation is

$$2\frac{\tau}{\rho}\left(\frac{\partial U}{\partial z}-\frac{kU}{1+kz}\right) = \text{dissipation.}$$

Therefore vanishing production of turbulent energy implies that

$$\partial U/\partial z = kU/(1+kz).$$

This condition then gives an absolute upper limit of one for $Ri_{f\,cr}$ for curved flows. The corresponding upper limit for Ri_{cr} is infinity. Unlike stratified flows, there is no second condition for the turbulent intensity to satisfy, therefore there is no *a priori* reason to expect $Ri_{f\,cr}$ to be much less than one as in the case of a stably stratified atmosphere. Nevertheless, laboratory observations of flows on curved surfaces by So & Mellor (1973) indicate that $Ri_{f\,cr} = 0{\cdot}215$, which is in excellent agreement with the value $Ri_{f\,cr} = 0{\cdot}25$ reported by Businger *et al.* (1971). This further supports the close analogy between the effects of buoyancy and streamline curvature.

From the above discussion of the critical flux Richardson number, it is clear that careful experimental work on curved shear flows and stratified flows is needed to determine more precisely the value of $Ri_{f\,cr}$.

8. Conclusions

A simple formula for the variation of the turbulent velocity scale with Richardson number has been derived from the Reynolds-stress equations by assuming that production of turbulent energy balances viscous dissipation. The resultant formula contains one free parameter, but it can be determined from data for unstratified flows. Consequently, there is no free constant in the formula. In the case of non-rotating curved flow or rotating plane flow, the formula can be shown to reduce to the Monin–Oboukhov formula for small Richardson number. Therefore, this provides direct evidence supporting the application of the Monin–Oboukhov formula to curved shear flows as suggested by Bradshaw (1969).

The present formula was found to be consistent with the limited data available on curved shear flows. However, more experimental work is needed to establish further the general validity of the present result.

By comparing the present result with that given by Mellor (1973), the curvature/buoyancy analogy was shown to be valid in the range

$$-0{\cdot}21 \leqslant Ri_f \leqslant 0{\cdot}21,$$

and the critical flux Richardson number beyond which turbulence cannot exist is in excellent agreement with that predicted by Mellor and the measurement of Businger *et al.* (1971). It was also found that good agreement between predictions and measurements of wind shear and temperature variance in an atmospheric surface layer is obtained in the range $-2 < Ri < 0{\cdot}30$.

Research supported by the Atmospheric Sciences Section, National Science Foundation, NSF Grant GA-33421. The author wishes to thank Prof. R. L. Peskin, who reviewed the manuscript and whose encouragement and support are deeply appreciated.

REFERENCES

Arya, S. P. S. 1972 *Quart. J. Roy. Met. Soc.* **98**, 264–273.

Bradshaw, P. 1969 *J. Fluid Mech.* **36**, 177–191.

Bradshaw, P. 1973 *AGARDograph*, no. 169.

Businger, J. A. 1966 *Proc. Symp. Arctic Heat Budget & Atmos. Circulation, Rand Corp., Santa Monica, Calif.*, pp. 305–322.

Businger, J. A., Wyngaard, J. C., Izumi, Y. & Bradley, E. 1971 *J. Atmos. Sci.* **28**, 181–189.

Dean, R. C. 1968 *A.S.M.E. Special Publ.* H-38.

Ellis, L. B. & Joubert, P. N. 1974 *J. Fluid Mech.* **62**, 65–84.

Eskinazi, S. & Yeh, H. 1956 *J. Aero. Sci.* **23**, 23–24.

Giles, J. A., Hays, A. P. & Sawyer, R. A. 1966 *Aero. Quart.* **14**, 201–215.

Görtler, H. 1959 *Ing. Arch.* **28**, 71–78.

Halleen, R. M. & Johnston, J. P. 1967 *Thermosci. Div., Mech. Engng Dept., Stanford University, Rep.* MD-18.

Johnson, D. S. 1959 *J. Appl. Mech.* **26**, 325–336.

Johnston, J. P. 1971 *AGARD Current Paper*, no. 93.

Klebanoff, P. S. 1955 *N.A.C.A. Rep.* no. 1247.

Kolmogorov, A. N. 1941 *C. R. Akad. Nauk. S.S.S.R.* **30**, 301–305. (Trans. *Turbulence, Classic Papers on Statistical Theory* (ed. S. K. Friedlander & L. Topper). Interscience, 1961.)

Laufer, J. 1954 *N.A.C.A. Rep.* no. 1174.

Lezius, D. K. & Johnston, J. P. 1971 *Thermosci. Div., Mech. Engng Dept., Stanford University, Rep* MD-29.

Lumley, J. L. & Panofsky, H. A. 1964 *The Structure of Atmospheric Turbulence.* Interscience.

Mellor, G. L. 1973 *J. Atmos. Sci.* **30**, 1061–1069.

Mellor, G. L. & Herring, H J. 1973 *A.I.A.A. J.* **11**, 590–599.

Mellor, G. L. & Yamada, T. 1974 *J. Atmos. Sci.* **31**, 1791–1806.

Monin, A. S. & Oboukhov, A. M. 1954 *Trudy Geofiz. Inst. AN S.S.S.R.* **24**, 163–187.

Moore, J. 1967 *M.I.T. Gas Turbine Lab. Rep.* no. 89.

Pandolfo, J. P. 1966 *J. Atmos. Sci.* **23**, 495–502.

Parr, O. 1963 *Ing. Arch.* **32**, 393–413.

Patel, V. C. 1969 *Aero. Res. Counc. R. & M.* no. 3599.

PRANDTL, L. 1929 *Sonderdruck aus Vortrage aus dem Gebiete der Aerodynamik und verwandter Gebiete.* Aachen. (Trans. *N.A.C.A. Tech. Memo.* no. 625.)

RASTOGI, A. K. & WHITELAW, J. H. 1971 *A.S.M.E. Paper*, no. 71-WA/FE-37.

RICHARDSON, L. F. 1920 *Proc. Roy. Soc.* A **97**, 354–373.

ROTTA, J. C. 1951*a* *Z. Phys.* **129**, 547–572.

ROTTA, J. C. 1951*b* *Z. Phys.* **131**, 51–77.

SCHMIDBAUER, H. 1936 *Luftfahrtforsch.* **13**, 160. (Trans. *N.A.S.A. Tech. Memo.* no. 791.)

SCHUBAUER, G. B. & KLEBANOFF, P. S. 1951 *N.A.C.A. Rep.* no. 1030.

SO, R. M. C. 1975 *Trans. A.S.M.E., J. Fluids Engng* (to appear).

SO, R. M. C. & MELLOR, G. L. 1972 *N.A.S.A. Contractor Rep.* no. 1940.

SO, R. M. C. & MELLOR, G. L. 1973 *J. Fluid Mech.* **60**, 43–62.

SO, R. M. C. & MELLOR, G. L. 1975 *Aero. Quart.* **26**, 25–40.

TOWNSEND, A. A. 1958 *J. Fluid Mech.* **3**, 361–372.

WATTENDORF, F. L. 1935 *Proc. Roy. Soc.* A **148**, 565–598.

WYNGAARD, J. C. & COTÉ, O. R. 1971 *J. Atmos. Sci.* **28**, 190–201.